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# ESTIMATING MIXTURE PROPORTIONS FOR COMPONENT WEIBULL DISTRIBUTIONS

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ABSTRACT

Mixture distributions characterize many physical measurements in which observable variates are generated from one of several component distributions. It is common to model such measurements as mixtures of normal distributions to estimate the model parameters. Often the primary interest in applications of this methodology is on the estimation of the mixing proportion(s). Estimation of crop proportions from remotely-sensed spectral measurements is an important application of proportion estimation in which the component distributions are not necessarily symmetric. In this paper mixtures of component Weibull distributions are investigated. Minimum distance estimation of the mixing proportion is the main focus of interest.

Key Words: Asymmetric Distributions; Identifiability; Maximum Likelihood; Minimum Distance

# 1. INTRODUCTION

Mixture distributions arise when the distribution of (possibly vector-valued) observable variates can be modeled as

$$f(x) = p_1f_1(x) + p_2f_2(x) + \ldots + p_mf_m(x), \tag{1.1}$$

where the m densities $f_1(x),\ldots,f_m(x)$ are referred to as component distributions and the $p_j$ $(j = 1,\ldots,m)$ are the mixing proportions $(\Sigma p_j = 1)$. An important application of mixture modeling occurs in satellite remote-sensing of agricultural characteristics; specifically, the use of spectral measurements of light intensity to determine crop types. Typically four or more spectral intensities (x) are taken on a portion of an agricultural field with the intent to estimate the proportions of corn, soybeans, or several other crops which are known to be grown in the region. The proportions are the mixture parameters $p_j$ and the spectral intensities or derived feature variables from each type of crop are represented by the component densities $f_j(x)$.

Figure 1 represents a histogram of truncated "peak greenness" variates from a 5x6 nautical mile segment of agricultural land in Minnesota. Peak greenness is derivable (Badhwar 1984) from spectral readings taken several times during the growing season. The histogram displays 200 peak greenness values, each of which represents one of two component crops: corn or soybeans. The assumption one makes about the component densities is crucial to the satisfactory estimation of the crop proportions.

[Insert Figure 1]

In order to illustrate this point, in Figure 2 we display graphs of two sets of (scaled) component distributions which yield mixture distributions similar to the histogram in Figure 1. In the upper mixture the component distributions are normal with $p_1 = .7$, and the standard deviation of the first component approximately twice that of the second component. In the lower mixture the component distributions are "reversed" $\chi^2(9)$ distributions with $p_1 = .5$ and both component standard deviations equal. These two mixture densities are so similar that histograms from the two mixture distributions would be visually indistinguishable; however, the mixture proportions are dramatically different for the two mixtures. Thus the ability to accommodate nonnormal component distributions is critical to the accurate estimation of mixture proportions.

[Insert Figure 2]

In this paper we investigate the estimation of the mixture proportion p in a mixture of two Weibull components

$$f(x) = pf_1(x) + (1 - p)f_2(x) \ , \qquad (1.2)$$

where each component distribution is a three-parameter Weibull density:

$$f(x) = \gamma_j\beta_j \ \{(x-\alpha_j^{-1})/\beta_j\}^{\gamma_j-1} \exp[-\{(x-\alpha_j)/\beta_j\}^{\gamma_j}] \qquad (1.3)$$

$$x \geq \alpha_j; \quad \beta_j, \gamma_j > 0 \ .$$

The three-parameter Weibull density is investigated in this study because of the flexibility it affords in allowing different distributional shapes and different threshold (truncation) points. In Section 2 basic properties of the Weibull distribution and the estimation of its parameters are discussed. In Section 3 simulation results compare minimum distance estimation of the proportion p from the Weibull mixture (1.2) with both maximum likelihood and minimum distance estimation of p under the assumption that the components in (1.2) are normal densities. A brief discussion of identifiability for Weibull distributions is contained in Section 4. Concluding remarks are made in Section 5.

## 2. THE WEIBULL DISTRIBUTION

The Weibull distribution has been widely used in recent years in the fields of reliability and quality control. Its popularity is largely due to its flexibility in describing distributions which are symmetric or skewed in either direction. The threshold parameter $\alpha$ in the Weibull density (1.3) allows the distribution to be shifted along the horizontal axis, an important property in the study of mixtures, $\beta$ serves as a scale parameter, and $\gamma$ determines the shape of the distribution. In Figure 3 Weibull densities for five sets of distributional parameters are graphed. From the figure it is apparent that the shape can vary dramatically; in particular, the ability of the Weibull distribution to be skewed to the right or to the left is clearly indicated.

[Insert Figure 3]

The cumulative distribution function corresponding to the three-parameter Weibull is given by the closed-form expression

$$F(x) = 1 - \exp[-\{(x-\alpha)/\beta\}^{\gamma}] \tag{2.1}$$

while the noncentral moments are given by (e.g., Dubey 1967)

$$\mu'_r = \Sigma_{k=0}^{r}\binom{r}{k}\alpha^{r-k}\beta^{k}\Gamma(k\gamma^{-1}+1). \tag{2.2}$$

From (2.2) it is readily found that

$$\mu = \alpha + \beta\Gamma(\gamma^{-1} + 1)$$

and (2.3)

$$\sigma^2 = \beta^2\{\Gamma(2\gamma^{-1} + 1) - \Gamma^2(\gamma^{-1} + 1)\}.$$

Dubey (1967) studied the relationship between the normal and Weibull distributions. He showed that when $\gamma$ is approximately 3.60232 the standardized skewness parameter $\mu_3/\{\mu_2^3\}^{1/2}$, where $\mu_j$ is the jth central moment, is zero indicating symmetry; moreover,

$$\sup_{-3\le t\le 3} |F_Z(t) - F_Y(t)| = .0078$$

where $F_Z$ denotes the cumulative distribution function of the random variable $Z \sim N(0,1)$ and Y is the standardized variate $Y = (X-\mu)/\sigma$ with $\mu$ and $\sigma^2$ denoting the mean and variance of the Weibull variate X, equations (2.3). If $\gamma < 3.60232$ the Weibull is skewed to the right, while if $\gamma > 3.60232$ it is skewed to the left.

Previous research in the estimation of Weibull mixtures includes that of Kao (1959), who proposed a graphical procedure for estimating the parameters of a Weibull mixture when one of the threshold parameters is assumed to be known and equal to zero. The estimation of the 6 remaining parameters is accomplished using a graphical procedure whose applicability to remote-sensing appears to be limited, although some of his estimation rules could be automated. Rider (1961) and Falls (1970) propose estimating the parameters of a mixture of two-parameter Weibull components using the method of moments. Falls' procedure involves estimating the mixing proportion p using a graphical procedure similar to that of Kao and then estimating the remaining parameters from the moment equations.

Maximum likelihood estimators are obtained by differentiating the log-likelihood function

$$\ln(L) = \Sigma \ln\{pf_1(x_i) + (1 - p)f_2(x_i)\}$$

with respect to each of the 7 model parameters, resulting in the likelihood equations $(j = 1,2)$

$$p = n^{-1}\Sigma f(1|x_i) \tag{2.4}$$

$$(\gamma_j-1)\Sigma f(j|x_i)(x_i-\alpha_j)^{-1} - \gamma_j\beta_j^{-\gamma_j}\Sigma f(j|x_i)(x_i-\alpha_j)^{\gamma_j-1} = 0 \tag{2.5}$$

$$\beta_j = [\{\Sigma(x_i-\alpha_j)^{\gamma_j} f(j|x_i)\}/\Sigma f(j|x_i)]^{1/\gamma_j} \tag{2.6}$$

$$\gamma_j = \{[\Sigma\{[(x_i-\alpha_j)/\beta_j]^{\gamma_j}-1\}\ell n[(x_i-\alpha_j)/\beta_j] f(j|x_i)]/\Sigma f(j|x_i)\}^{-1} \tag{2.7}$$

where $f(j|x) = p_j f_j(x)/f(x)$, $f_j(x)$ denotes the jth component density, $f(x)$ is the mixture density, $p_1 = p$, and $p_2 = 1-p$, and n is the sample size. Solving this set of equations for the maximum likelihood estimators is difficult due largely to equations (2.5) which are not in fixed-point form.

Olssen (1979) discusses the numerical maximization of the likelihood function when the data are grouped into equally-spaced intervals and the first component threshold parameter is zero. He does not consider the more general framework of interest in this work; namely, two (possibly) nonzero threshold parameters and nongrouped data. Looney and Bargmann (1982) also suggest a procedure which can be used with grouped data. Preliminary estimates of p, $\alpha_1$, $\alpha_2$, $\beta_1$, and $\beta_2$ are found for a grid of values of $\gamma_1$ and $\gamma_2$: all possible combinations of

$$\gamma_j = 1/5,\ 1/4,\ 1/3,\ 1/2,\ 2/3,\ 1.0,\ 1.5,\ 2.0,\ 3.0\ 4.0\ 5.0.$$

For each $(\gamma_1, \gamma_2)$ pair, the likelihood equations for grouped data are then solved for the remaining 5 parameters using a small number of iterations and the $(\gamma_1, \gamma_2)$ pair for which ln(L) is maximized is identified. With $\gamma_1$ and $\gamma_2$ fixed at these values, maximum likelihood estimation for the remaining 5 parameters is then carried through to convergence. The Looney and Bargmann procedure for finding maximum likelihood estimates seems overly restrictive with respect to the selection of possible values of the shape parameter, while expansion of the search procedure to allow for more shape parameter values could be prohibitive because of the wide range of possible values.

Because of the computational difficulties of directly solving the likelihood equations, we consider the use of minimum distance (MD) estimation, first introduced by Wolfowitz (1957), for simultaneously estimating the 7 parameters in the Weibull mixture. Woodward, et al. (1984) recently studied the use of MD estimation in the normal mixture model. They showed that MD estimation is easy to implement for mixtures of normal distributions and that MD estimators are superior to maximum likelihood estimators under symmetric departures from component normality. Since our use of Weibull components is due to the flexibility which they introduce into the model rather than an underlying theoretical justification, an estimation procedure which is robust to departures from basic distributional assumptions is highly desirable.

The minimum distance estimator of the parameter $\theta$ (possibly vector-valued) is defined to be that value of $\theta$ which minimizes the distance between $H_\theta$ and $F_n$, where $\mathcal{H} = \{H_\theta: \theta \varepsilon \Omega\}$ denotes a family of distributions depending on $\theta$, and $F_n$ denotes the empirical distribution function; i.e., $F_n(x) = k/n$ where k is the number of observations less than or equal to x. The family of distributions $\mathcal{H}$ is referred to as the projection model, where in this case $\theta = (p, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)$, and $H_\theta(x)$ is the distribution function associated with a mixture of two Weibull components:

$$H_\theta(x) = p\{1-\exp[-\{(x-\alpha_1)/\beta_1\}^{\gamma_1}]\} + (1-p)\{1-\exp[-\{(x-\alpha_2)/\beta_2\}^{\gamma_2}]\}. \quad (2.8)$$

Note that in contrast to the situation in which the projection model is taken to be the mixture of two normals, $H_\theta(x)$ in (2.8) has a closed form expression.

The choice of distance function to be used to measure the distance between two distributions is a topic of current interest in the field of MD estimation. Woodward, et al. (1984) used the Cramer-von Mises distance, $W^2$, given by

$$W^2 = \int_{-\infty}^{\infty} [G_1(x) - G_2(x)]^2 dG_2(x) \ , \qquad (2.9)$$

where $G_1$ and $G_2$ are two distribution functions, and we have chosen to use this distance measure in the current study. The Cramer-von Mises distance between an empirical distribution function $G_1 = F_n$ and a projection family member $G_2 = H_\theta$ can be simplified to yield the following expression:

$$W^2 = (12n)^{-1} + \Sigma\{H_\theta(Y_i) - (i-0.5)/n\}^2 \ , \qquad (2.10)$$

where $Y_i$ denotes the ith sample order statistic. Since $H_\theta$ exists in closed form, the MD estimator of the mixture model parameters is easily obtained by using nonlinear least squares techniques to minimize (2.10). We perform this minimization with IMSL (1982) subroutine ZXSSQ which uses Marquardt's (1963) algorithm.

## 3. SIMULATION RESULTS

From the discussion of the previous section, it appears that minimum distance techniques have potential value for estimating a mixture of three-parameter Weibull distributions, especially in terms of computational convenience. In this section we discuss the results of a computer simulation which was designed to evaluate the numerical capabilities of this method.

In order to assess the merit of using "symmetry-flexible" projection families, the focus of this simulation is on the comparison of Weibull-based MD estimation with normal-based procedures. Samples are generated from a variety of component distributions: normal, double exponential, t(4), t(2), and $\chi^2(9)$. Normal-based procedures should perform better than Weibull-based procedures when the mixture contains normal or unimodal, symmetric components; however, if the Weibull techniques are to be useful, they must produce reasonable results in this situation. In addition, when the component distributions are skewed, the Weibull-based procedures should result in more accurate proportion estimates. No simulation results are reported for mixtures of component Weibull distributions because we propose the Weibull as a flexible projection family and not necessarily as the true component distribution.

For ease of presentation the two component distributions differ from each other only by a location shift. The mixture distributions have mixing proportions of .25, .50, and .75 with varying degrees of separation between the two component distributions. "Overlap" as defined by Woodward, et al. (1984) is a quantification of this separation. It is defined as the probability of misclassification using the rule:

Classify an observations x as:

population 1 if $x < x_c$

population 2 if $x \geq x_c$

where without loss of generality population 1 is assumed to be centered to the left of population 2, and where $x_c$ is the unique point between $\mu_1$ and $\mu_2$ such that $pf_1(x_c) = (1-p)f_2(x_c)$. The current study is based on overlaps of .03 and .10.

In Figure 4 the mixture densities associated with normal components are displayed. For each mixture, the weighted components $pf_1(x)$ and $(1-p)f_2(x)$ are also shown. The densities for $p = .75$ are not displayed here since their shapes can be inferred from those at $p = .25$. Likewise, parameter estimation for $p = .75$ is not included in the results of the simulations for the mixtures of symmetric component distributions. For comparison purposes mixture densities associated with the mixtures of $\chi^2(9)$ components are displayed in Figure 5. Note that although we refer to a mixture of $\chi^2(9)$ distributions, they are actually "shifted" chi-squares; i.e., the threshold parameter for one of the component distributions differs from zero.

[Insert Figures 4 and 5]

The simulation results are based on 100 samples of size $n = 200$ from mixtures of the various component distributions. Uniform random numbers were generated for use in obtaining variates for component distributions from the IMSL (1982) multiplicative congruential uniform generator GGUBS. Normal deviates were generated using IMSL subroutine GGNPM, which uses the polar method, and $t(\nu)$ variates were based on ratios of independent normal and $(\chi^2(\nu)/\nu)^{1/2}$ variates, the chisquare variates obtained from IMSL subroutine GGCHS. Double exponential components were generated as $\ln(U)$, where U is a uniform (0,1) variate, with the positive or negative sign randomly assigned. Observations from standardized component distributions were randomly assigned to population 1 or population 2 depending on whether an independent uniform variate was less than or greater than p, respectively. The standardized components were then scaled and shifted to conform to the assigned distribution. All computations were performed on a CDC 6600 at Southern Methodist University.

The normal component models were generated with $\mu_1 = 0$, $\sigma_1^2 = \sigma_2^2 = 1$, and $\mu_2$ positioned so that the desired overlap was obtained. Likewise, the mixtures for the other symmetric component distributions were generated with one of the components centered at 0 and the other one shifted to the right by a sufficient amount to produce the desired overlap. The chisquare mixtures were generated with one of the threshold parameters fixed at zero and the other one shifted so that the desired overlap was obtained. For maximum likelihood and minimum distance estimation with normal component distributions, the natural constraints $\sigma_1^2 > 0$, $\sigma_2^2 > 0$, and $0 \le p \le 1$ were imposed. Similarly, for minimum distance estimation using Weibull components the natural constraints $\beta_1 > 0$, $\beta_2 > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$, and $0 \le p \le 1$ were imposed.

For each of the simulated samples, three sets of parameter estimates were obtained:

(1) ML estimates based on a mixture of normal components (MLEN)

(2) MD estimates based on a mixture of normal components (MDEN)

(3) MD estimates based on a mixture of Weibull components (MDEW).

Starting values for the iteration routines were calculated using the robust estimation technique outlined in Woodward et al. (1984). Basically this technique chooses p = .1, .2,..., or .9 depending on which value maximizes a standardized measure of the difference between the sample population medians. Once p is selected robust estimates of the samples means and variances are obtained for population 1 from the smallest $n_1 = np$ (rounded to the nearest integer) observations and for population 2 from the largest $n_2 = n - n_1$ observations. For the mixture of normal components these initial estimates are substituted for the model parameters. A similar procedure is used for the mixture of Weibull components except that (a) the shape parameters are always initially set at $\gamma_j = 3.6$ (symmetry) and (b) the initial truncation and scale parameters are obtained by inserting the robust estimates of the mean and variance in the left sides of equations (2.3) and solving them simultaneously (using $\gamma_j = 3.6$).

Although the MLEN and MDEN provide estimates of all 5 of the parameters of the normal mixture model and the MDEW produces estimates for all 7 parameters in the Weibull mixture model, only the results for the estimation of p will be shown. The mixing proportion is the parameter of primary interest and when the assumed component distributions are not the true component distributions which generate the data the remaining parameter estimates often do not have a meaningful interpretation.

The results of the simulations are displayed in Table 1 to 6. For a given simulation model and estimation procedure, we obtain an estimate $\hat{p}$ of p, defined by

$$\hat{p} = n_s^{-1} \Sigma \hat{p}_i ,$$

where $\hat{p}_i$ is the estimate of p for the ith sample and $n_s = 100$ is the number of samples. Estimates of the bias and mean squared error (mse) of $\hat{p}$ are

$$\widehat{\text{bias}} = n_s^{-1} \Sigma(\hat{p}_i - p) = \hat{p} - p$$

$$\widehat{\text{mse}} = n_s^{-1} \Sigma(\hat{p}_i - p)^2 \quad .$$

As would be expected, all estimators at the .03 overlap perform better than at the .10 overlap. Tables 1 to 4 indicate that when the true component distributions are symmetric, all the estimators perform best when $p = .50$. In general MLEN performs best for the normal mixtures in Table 1 while MDEN is superior for the mixtures of nonnormal symmetric components in Tables 2-4. MDEW performs well at $p = .50$ but is less satisfactory than assuming a normal component model when the simulation component models are symmetric and $p = .25$, although for many of the simulation models it performs comparably to MLEN. Overall Tables 1-4 suggest that minimum distance estimation with Weibull component models is no less satisfactory than the normal-based procedures for estimating the mixture proportion in mixtures of symmetric distributions.

[Insert Table 1-4]

Figure 4 provides a visual explanation for the performance of the estimation schemes when the components are symmetric. The mixture distributions for $p = .50$ display both bimodality and symmetry whereas the mixture for $p = .25$ and an overlap of .10 does not even display bimodality. It is not surprising that the estimation procedures perform poorly for this latter mixture.

Tables 5 and 6 display the simulation results for mixtures of $\chi^2(9)$ component distributions. The negatively-skewed ("reversed") chisquare components used to generate the results reported in Table 6 are mirror-images (around the origin) of the distributions shown in Figure 5. Minimum distance estimation with Weibull component distributions performs as well or better than the normal estimation schemes for all three mixing proportions and both component separations. The MDEW is especially satisfactory for the simulations with larger separation (overlap = .03). A visual explanation for the performance of the MDEW can be obtained from an examination of Figure 5. With large separation both the bimodality and the asymmetry are evident in the mixture. Minimum distance estimation with Weibull components thus can be expected to perform better than normal-based estimation procedures when the component distributions are sufficiently separated and asymmetry is pronounced. Notice that the chisquare mixtures with p = .50 and overlap = .10 possess densities similar to those given in Figure 2. Since the normal-based estimation procedures assume symmetric components, the large bias shown in the MLEN and MDEN is understandable.

## 4. PRACTICAL NONIDENTIFIABILITY OF THE THREE-PARAMETER WEIBULL DISTRIBUTION

Analyses of several of the simulations reported in the last section revealed an unexpected complication to the fitting of three-parameter Weibull distributions. Although we concentrate only on the estimation of the mixing parameter p in this paper, our simulation programs detail summary information on the estimation of all seven of the parameters in the Weibull mixture. For several samples the parameter estimates of $\alpha_j$ were large and negative while those for $\beta_j$ were large and positive, sometimes each was greater than 1000 in magnitude. These parameter values were associated with a $\gamma_j$ smaller in magnitude than $\alpha_j$ and $\beta_j$ but substantially larger than 3.6. Although these parameter values appeared to be unacceptably poor, plots of the associated Weibull densities were consistent with the data, with only very small probability being associated with the interval between $\alpha_j$ and 0.

Further investigation of these sample estimates indicated that whenever two Weibull models with very different parameter sets possessed similar densities, the two distributions were characterized by

(a) left skewness

(b) similar ratios of scale to shape parameters

(c) similar sums of threshold and scale parameters.

It can be shown analytically that two models sharing the three properties mentioned above will have similar modes, heights at the modes, skewness, and variances. In Figure 6, we display four Weibull densities with parameter sets satisfying the above three conditions. Note that for all four densities, $\gamma > 3.6$, $\beta/\gamma = 4$, and $\alpha + \beta = 30$. The densities associated with the second, third, and fourth parameter sets are almost indistinguishable.

[Insert Figure 6]

We conclude from these observations that the three-parameter

Weibull suffers from practical nonidentifiability. The nonidentifiability is theoretically impermissible since Weibull distributions with unequal threshold parameters ($\alpha_1 < \alpha_2$, say) cannot have identical densities. These densities would obviously differ between $\alpha_1$ and $\alpha_2$ where $f_1(x) > 0$ and $f_2(x) = 0$. Our immediate concern is the extent to which this practical lack of identifiability affects parameter estimation in the Weibull mixture model. If only proportion estimates are desired, there seems to be no real effect. It is clear, however, that component Weibull parameter estimates can be very misleading.

## 5. CONCLUDING REMARKS

Three-parameter Weibull distributions offer the flexibility of representing many different shapes of probability distributions, both symmetric and asymmetric. Prior to the successful implementation of this methodology several remaining research problems must be resolved.

The simulations reported in Section 3 indicate that minimum distance estimation of the mixing proportion using three-parameter Weibull distributions as the projection family can provide acceptable parameter estimates if the separation between the component distributions is adequate. In such situations both symmetric and asymmetric distributions can be modeled, the latter ability offering superior performance to normal-based procedures when asymmetry is pronounced. Additional simulations (not reported) were performed on component distributions which differ in both location and scale. The results were in general agreement with those reported in Section 3. Simulations were also performed on mixtures of $\chi^2(9)$ distributions for which one component was positively-skewed and one was negatively-skewed. These mixtures tended to mask the asymmetry and the overall conclusions drawn from these simulations were similar to those for symmetric components.

Further research is needed to assess the estimation efficacy of model parameters other than the mixture proportion. When using the 3-parameter Weibull distribution as a projection family, all the remaining parameters may have little meaning with respect to the true underlying component distributions. If so, the practical nonidentifiability of the three-parameter Weibull distribution may be of no consequence; however, when the true underlying distribution closely approximates the Weibull, indeed if the Weibull is the true theoretical component distribution, then this practical nonidentifiability is of concern and techniques must be devised to compensate for its presence when one desires to estimate all of the model parameters.

In several of the simulation results reported in Section 3 and in Woodward, et al. (1984) the starting values were more accurate and/or precise than the maximum likelihood and minimum distance estimators. Another instance where crude initial parameter choices appeared to be as good as maximum likelihood estimates for mixtures of normal or Weibull components is the examples in Olssen (1979). This ability of simple robust procedures to equal or outperform asymptotically optimal estimation techniques warrants further investigation.

Finally, there is much work to be done in extending these results to mixtures of more than two component distributions. The Weibull distribution coupled with minimum distance estimation is technically restricted only by the ability to find the minimum of equation (2.10) in a computationally efficient manner. Since MDEW tended to require more computer time than the normal-based estimators, computational efficiency is an important consideration in higher-dimensional modeling.

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Table 1. Comparison of Proportion Estimation Techniques, Mixtures of Normal Components

| | Overlap=.10 | | | Overlap = .03 | | |
|---|---|---|---|---|---|---|
| | $\hat{p}$ | $\widehat{bias}$ | $\widehat{mse}$ | $\hat{p}$ | $\widehat{bias}$ | $\widehat{mse}$ |
| | | | (a) p = .25 | | | |
| MLEN | .27 | .02 | .0234 | .25 | .00 | .0015 |
| MDEN | .31 | .06 | .0450 | .26 | .01 | .0022 |
| MDEW | .33 | .08 | .0449 | .29 | .04 | .0105 |
| Starts | .30 | .05 | .0089 | .30 | .05 | .0053 |
| | | | (b) p = .50 | | | |
| MLEN | .50 | .00 | .0200 | .50 | .00 | .0017 |
| MDEN | .51 | .01 | .0209 | .50 | .00 | .0019 |
| MDEW | .49 | -.01 | .0281 | .51 | .01 | .0063 |
| Starts | .51 | .01 | .0074 | .51 | .01 | .0055 |

Table 2. Comparison of Proportion Estimation Techniques, Mixtures of Double Exponential Components

| | Overlap = .10 | | | Overlap = .03 | | |
|---|---|---|---|---|---|---|
| | $\hat{p}$ | bi$\hat{a}$s | m$\hat{s}$e | $\hat{p}$ | bi$\hat{a}$s | m$\hat{s}$e |
| | | | (a) p = .25 | | | |
| MLEN | .42 | .17 | .0905 | .28 | .03 | .0054 |
| MDEN | .27 | .02 | .0078 | .29 | .04 | .0050 |
| MDEW | .47 | .22 | .0663 | .40 | .15 | .0272 |
| Starts | .33 | .08 | .0115 | .33 | .08 | .0099 |
| | | | (b) p = .50 | | | |
| MLEN | .50 | .00 | .0277 | .50 | .00 | .0015 |
| MDEN | .50 | .00 | .0040 | .50 | .00 | .0015 |
| MDEW | .49 | -.01 | .0092 | .50 | .00 | .0015 |
| Starts | .51 | .01 | .0066 | .50 | .00 | .0021 |

Table 3. Comparison of Proportion Estimation Techniques, Mixtures of t(4) Components

| | Overlap = .10 | | | Overlap = .03 | | |
|---|---|---|---|---|---|---|
| | $\hat{p}$ | $\hat{bias}$ | $\hat{mse}$ | $\hat{p}$ | $\hat{bias}$ | $\hat{mse}$ |
| | | | (a) $p = .25$ | | | |
| MLEN | .35 | .10 | .0767 | .28 | .03 | .0073 |
| MDEN | .28 | .03 | .0189 | .26 | .01 | .0013 |
| MDEW | .41 | .16 | .0574 | .34 | .09 | .0137 |
| Starts | .33 | .08 | .0111 | .33 | .08 | .0091 |
| | | | (b) $p = .50$ | | | |
| MLEN | .52 | .02 | .0580 | .50 | .00 | .0018 |
| MDEN | .51 | .01 | .0071 | .50 | .00 | .0015 |
| MDEW | .50 | .00 | .0070 | .49 | -.01 | .0022 |
| Starts | .50 | .00 | .0066 | .50 | .00 | .0021 |

Table 4. Comparison of Proportion Estimation Techniques, Mixtures of t(2) Components.

| | Overlap = .10 | | | Overlap = .03 | | |
|---|---|---|---|---|---|---|
| | $\hat{p}$ | $\hat{bias}$ | $\hat{mse}$ | $\hat{p}$ | $\hat{bias}$ | $\hat{mse}$ |
| | | | (a) p = .25 | | | |
| MLEN | .50 | .25 | .1876 | .33 | .08 | .0348 |
| MDEN | .29 | .04 | .0203 | .27 | .02 | .0018 |
| MDEW | .47 | .22 | .0687 | .35 | .10 | .0144 |
| Starts | .30 | .05 | .0091 | .35 | .10 | .0123 |
| | | | (b) p = .50 | | | |
| MLEN | .53 | .03 | .1446 | .51 | .01 | .0373 |
| MDEN | .49 | -.01 | .0055 | .51 | .01 | .0014 |
| MDEW | .50 | .00 | .0052 | .50 | .00 | .0012 |
| Starts | .51 | .01 | .0083 | .49 | -.01 | .0020 |

Table 5. Comparison of Proportion Estimation Techniques, Mixtures of Positively-Skewed Chisquare (9) Components

| | Overlap = .10 | | | Overlap = .03 | | |
|---|---|---|---|---|---|---|
| | $\hat{p}$ | $\widehat{bias}$ | $\widehat{mse}$ | $\hat{p}$ | $\widehat{bias}$ | $\widehat{mse}$ |
| | | | (a) p = .25 | | | |
| MLEN | .53 | .28 | .2190 | .17 | -.08 | .0102 |
| MDEN | .47 | .22 | .1712 | .16 | -.09 | .0095 |
| MDEW | .47 | .22 | .1017 | .31 | .06 | .0131 |
| Starts | .46 | .21 | .1009 | .26 | .01 | .0029 |
| | | | (b) p = .50 | | | |
| MLEN | .25 | -.25 | .0656 | .40 | -.10 | .0111 |
| MDEN | .27 | -.23 | .0620 | .41 | -.09 | .0089 |
| MDEW | .43 | -.07 | .0232 | .51 | .01 | .0043 |
| Starts | .49 | -.01 | .0080 | .47 | -.03 | .0052 |
| | | | (c) p = .75 | | | |
| MLEN | .47 | -.28 | .0830 | .63 | -.12 | .0198 |
| MDEN | .45 | -.30 | .1042 | .61 | -.14 | .0238 |
| MDEW | .50 | -.25 | .1029 | .70 | -.05 | .0118 |
| Starts | .69 | -.06 | .0109 | .66 | -.09 | .0125 |

Table 6. Comparison of Proportion Estimation Techniques, Mixtures of Negatively-Skewed Chisquare (9) Components.

| | Overlap = .10 | | | Overlap = .03 | | |
|---|---|---|---|---|---|---|
| | $\hat{p}$ | $\widehat{bias}$ | $\widehat{mse}$ | $\hat{p}$ | $\widehat{bias}$ | $\widehat{mse}$ |
| | | | (a) p = .25 | | | |
| MLEN | .55 | .30 | .0926 | .37 | .12 | .0188 |
| MDEN | .56 | .31 | .1111 | .38 | .13 | .0215 |
| MDEW | .47 | .22 | .0723 | .30 | .05 | .0074 |
| Starts | .32 | .07 | .1117 | .33 | .08 | .0117 |
| | | | (b) p = .50 | | | |
| MLEN | .73 | .23 | .0641 | .58 | .08 | .0089 |
| MDEN | .73 | .23 | .0600 | .58 | .08 | .0081 |
| MDEW | .62 | .12 | .0269 | .52 | .02 | .0027 |
| Starts | .50 | .00 | .0069 | .50 | .00 | .0048 |
| | | | (c) p = .75 | | | |
| MLEN | .49 | -.26 | .1988 | .83 | .08 | .0072 |
| MDEN | .58 | -.17 | .1563 | .83 | .08 | .0105 |
| MDEW | .63 | -.12 | .0671 | .78 | .03 | .0030 |
| Starts | .56 | -.19 | .0853 | .72 | -.03 | .0037 |

Figure 1.    Histogram of Peak Greenness Values

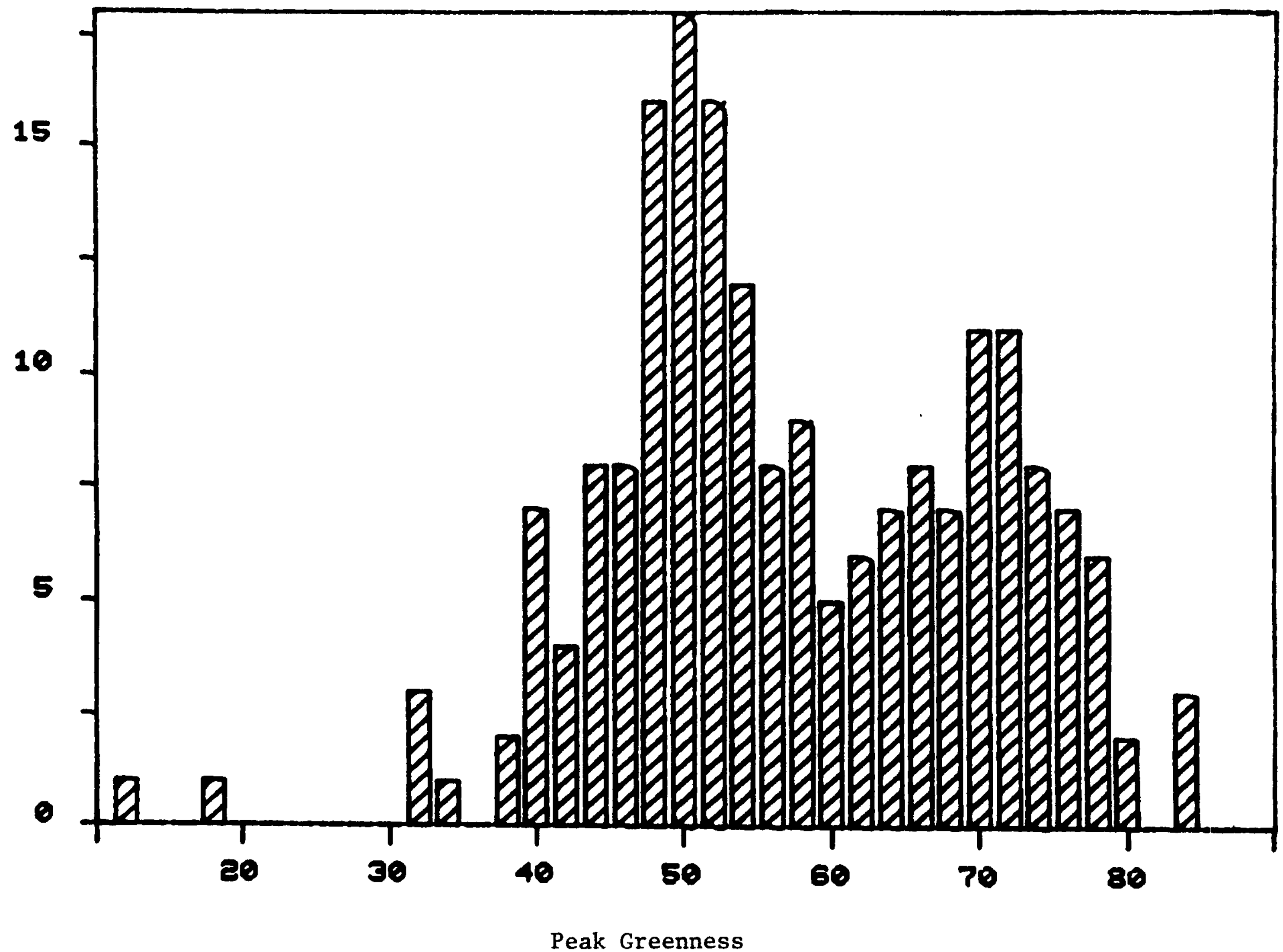
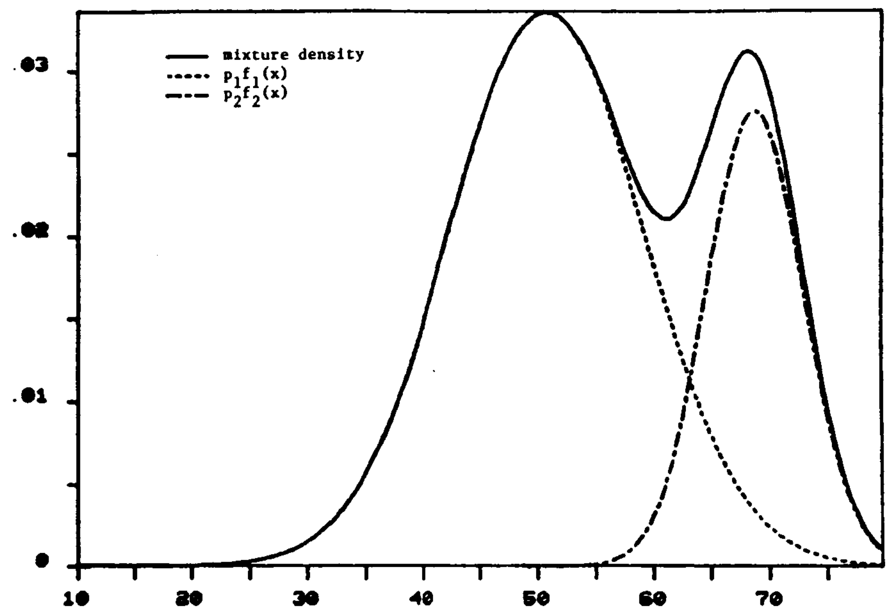
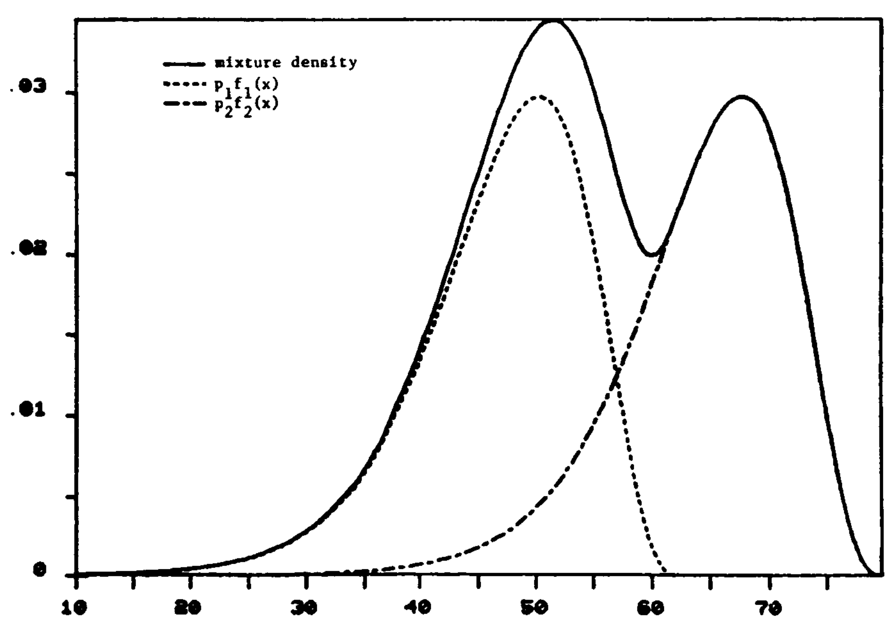


Figure 2. Two Mixture Distributions




(a) Mixture of Normal Components ($p_1$=.7)




(b) Mixtures of Reverse $\chi^2(9)$ Components ($p_1$=.5)

Figure 3.    Weibull Densities

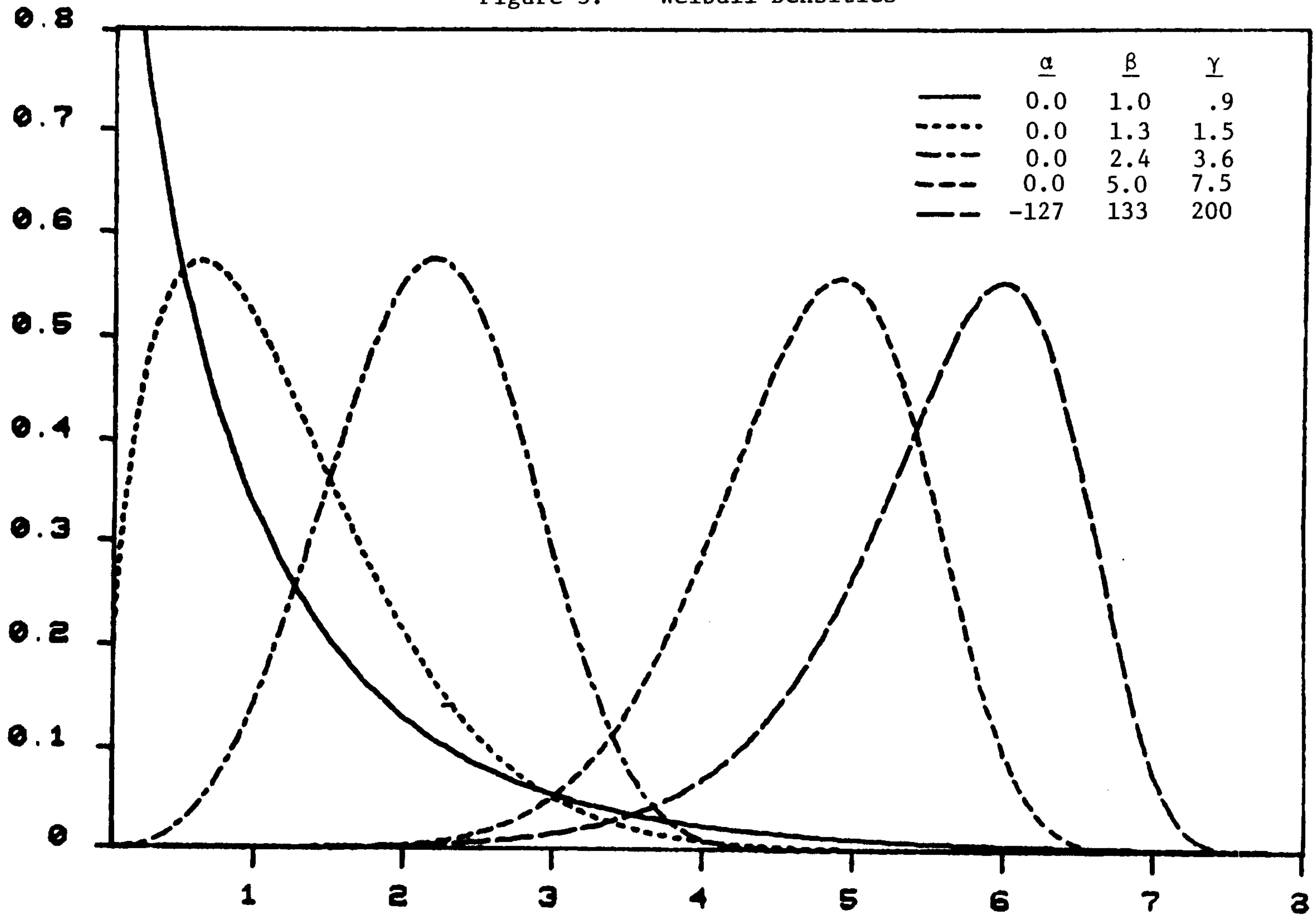
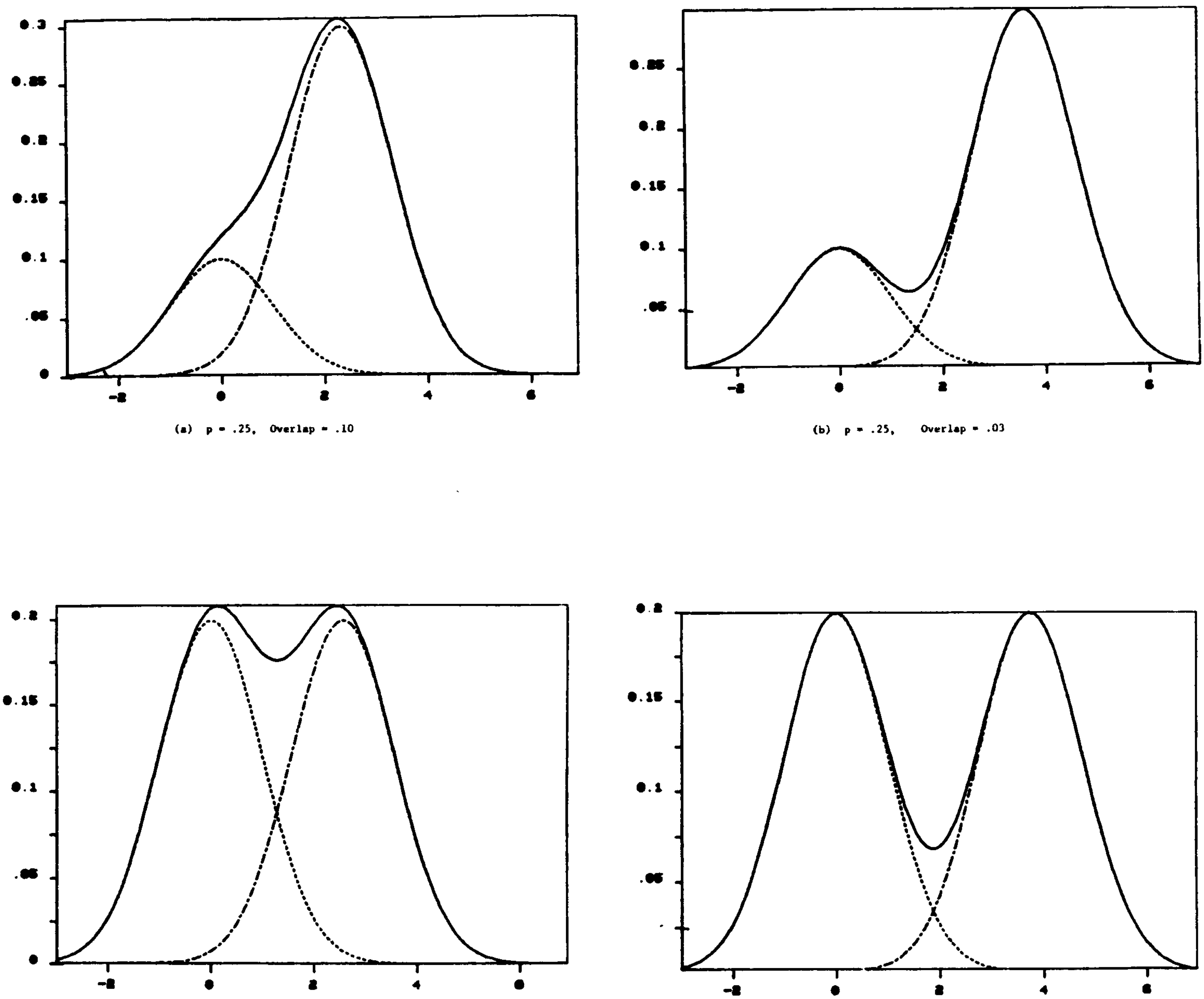


Figure 4. Mixture Densities with Normal Components




(a) p = .25, Overlap = .10

(b) p = .25, Overlap = .03

(c) p = .50, Overlap = .10

(d) p = .50, Overlap = .03

Figure 5. Mixtures Densities with $\chi^2(9)$ Components

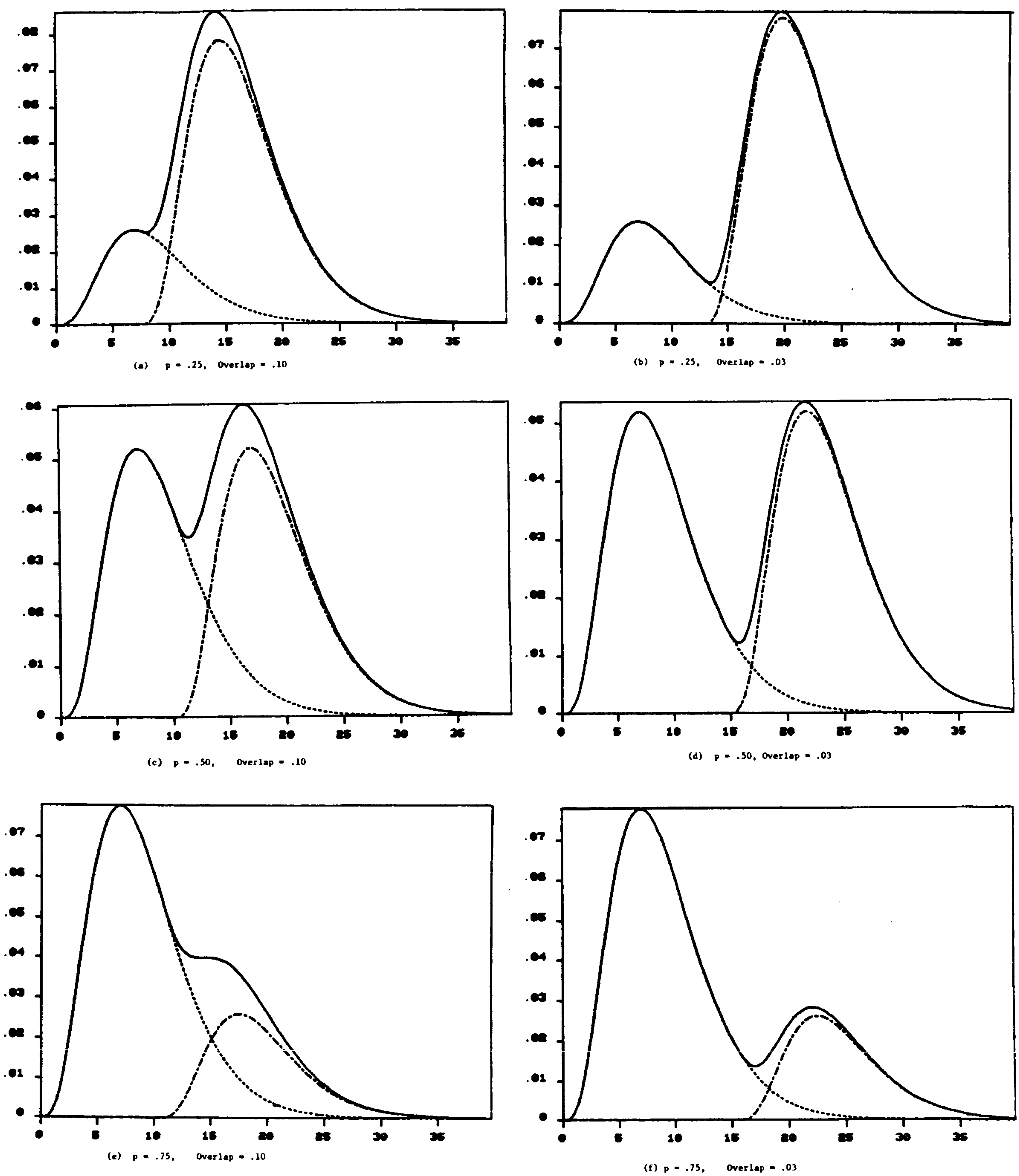


Figure 6. Practical Non-Identifiability of the Three-Parameter Weibull